About the solution set of the null controllability problem for the chain of integrators system

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Introduction

In this work we describe the solution set of the null controllability (SNC) problem for the chain of integrators system (0.1). We reduce the controllability problem to a Hausdorff moment problem (HMP) which is treated with help of the V.P. Potapov's Fundamental Matrix Inequality (FMI) method. An example of continuous, explicit solution of the SNC problem is given.

Notations. We use $\mathbb{R}^n$,$\mathbb{C}$ to denote the sets of n–dimensional Euclidean space ($\mathbb{R}$ is the set of real numbers) and complex numbers, respectively. We will use $\mathcal{C}_L^f$ to denote the set of all functions $f : 0 \leq f(\tau) \leq L,\ \tau \in [a, b]$. The symbol $\mathcal{M}[a, b]$ stands for the set of all nonnegative measures on $[a, b]$. $\bar{z}$ and w^* denote the complex conjugate of the number z and function w, respectively.

Statement of the problem. We consider the following controllable system,

$$\dot{x} = Ax + bu,\ x(0) = x_0, |u| \leq 1 \tag{0.1}$$

where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{0.2}$$

Here $x \in \mathbb{R}^n$, obviously A is a $n \times n$ matrix. It is required to describe the set of *all* controls u such that $|u(t)| \leq 1$, $t \in [0, \theta]$, for some θ: $x(\theta) = 0$.
The case $\theta \to \min$ which represents the time optimal control, was considered in [1].

Now we write some notions about moment problems which are crucial for the present work:
L Markov moment problem in finite interval [a,b]
Let be given a sequence of real numbers $\{c_j\}_{j=0}^k$. Find the set of functions $f : f \in \mathcal{C}_L^f$ such that the relation

$$c_j = \int_a^b \tau^j f(\tau) d\tau,\ j \in \{0, \cdots, k\}. \tag{0.3}$$

holds. We use $\mathcal{C}_L^f(\{c_j\}_{j=0}^k)$ to denote the set of solutions of (0.3). Remark $\mathcal{C}_L^f(\{c_j\}_{j=0}^k) \subseteq \mathcal{C}_L^f$.

The finite Hausdorff moment problem. The classical power moment problem for an interval $[a, b]$ is stated as follows: Let be given a finite sequence of real numbers $\{s_j\}_{j=0}^k$, such that

$$s_j = \int_a^b \tau^j \sigma(d\tau),\ j \in \{0, \cdots, k\}. \tag{0.4}$$

It is required to find the set of measures $\sigma : \sigma \in \mathcal{M}[a, b]$ such that (0.4) holds. We use $\mathcal{M}([a, b], \{s_j\}_{j=0}^k)$ to denote the set of solutions of (0.4). Remark that $\mathcal{M}([a, b], \{s_j\}_{j=0}^k) \subseteq \mathcal{M}[a, b]$.

Relation between the L–Markov power moment and the finite Hausdorff moment problem
The treatment of the L–Markov moment problem is usually connected to the problem of

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finding a holomorphic function, $z \in \mathbb{C} \setminus [a, b]$

$$c(z) = \int_a^b \frac{f(\tau)}{\tau - z} d\tau, \; f \in \mathcal{C}_L^f. \qquad (0.5)$$

In terms of the asymptotic expansion

$$\begin{aligned} \int_a^b \frac{f(\tau)}{\tau - z} d\tau &= -\frac{1}{z} \int_a^b \left(1 - \frac{\tau}{z}\right)^{-1} f(\tau) d\tau \\ &= -\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_a^b \tau^j f(\tau) d\tau \\ &= -\sum_{j=0}^{\infty} \frac{c_j^f}{z^{j+1}}, \qquad (0.6) \end{aligned}$$

it is required to find the set of functions f : $f \in \mathcal{C}_L^f$ such that $c_j^f = c_j$, $j \in \{0, \ldots, k\}$, that is $f \in \mathcal{C}_L^f(\{c_j\}_{j=0}^k)$. Here $c_j^f = \int_a^b \tau^j f(\tau) d\tau$, $f \in \mathcal{C}_L^f$ and c_j is number of a given sequence of numbers $\{c_j\}_{j=0}^k$.
In a similar way, a holomorphic function defined in $z \in \mathbb{C} \setminus [a, b]$,

$$s(z) = \int_a^b \frac{\sigma(d\tau)}{\tau - z}, \qquad (0.7)$$

called the associated function or Stieltjes transform of σ ($\sigma \in \mathcal{M}[a, b]$), is usually connected to the problem (0.4). Its asymptotic expansion

$$\begin{aligned} \int_a^b \frac{\sigma(d\tau)}{\tau - z} &= -\frac{1}{z} \int_a^b \left(1 - \frac{\tau}{z}\right)^{-1} \sigma(d\tau) \\ &= -\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_a^b \tau^j \sigma(d\tau) \\ &= -\sum_{j=0}^{\infty} \frac{s_j^\sigma}{z^{j+1}}, \qquad (0.8) \end{aligned}$$

reduces the considered moment problem to the problem of finding a set of σ such that $s_j^\sigma = s_j$, $j \in \{0, \ldots, k\}$. Here $s_j^\sigma = \int_a^b \tau^j \sigma(d\tau)$, $\sigma \in \mathcal{M}[a, b]$. That is, we find the set of measures $\sigma \in \mathcal{M}([a, b], \{s_j\}_{j=0}^k)$.

Let us remark that the Stieltjes transform determines the measure σ uniquely.

The *relation* between the problem (0.5) and (0.7) is given by the equation, (see [4])

$$\int_a^b \frac{\sigma(d\tau)}{z - \tau} = \frac{1}{z - a} \text{Exp} \left(\frac{1}{L} \int_a^b \frac{f(\tau) d\tau}{z - \tau} \right). \qquad (0.9)$$

The asymptotic expansion of the left and right sides of (0.9) gives

$$\begin{aligned} &\frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots \\ &\quad = \frac{1}{z - a} \text{Exp} \left[\frac{1}{L} \left(\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots \right) \right]. \qquad (0.10) \end{aligned}$$

The equality (0.10) turns into the following explicit relation between c_j and s_j, $j \in \{0, \cdots, k\}$ (see [5]), (here for simplicity, $a = 0$)

$$s_0 = 1, \; s_1 = \frac{c_0}{L}, \; s_2 = \frac{c_1}{L} + \frac{c_0^2}{2L^2},$$

$$s_3 = \frac{c_2}{L} + \frac{c_0 c_1}{L^2} + \frac{c_0^3}{6L^3},$$

$$\begin{aligned} s_{j+1} &= \frac{1}{(j+1)! L^{j+1}} \\ &\cdot \begin{vmatrix} c_0 & -L & \cdots & 0 \\ 2c_1 & c_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ jc_{j-1} & (j-1)c_{j-2} & \cdots & -jL \\ (j+1)c_j & jc_{j-1} & \cdots & c_0 \end{vmatrix} \\ &= \frac{c_j}{L} + \frac{c_0 c_{j-1}}{L^2} + \cdots, \quad (j \geq 0). \qquad (0.11) \end{aligned}$$

Using the bijective relation (0.11), the L–Markov moment problem can be solved in terms of the $[a, b]$–Hausdorff moment problem. We carry out the treatment of the last problem with help of the Potapov's FMI approach, (see [2], [3]). Let be remarked that in [2] and [3] an explicit solution of the nondegenerate matrix version of the Hausdorff matrix moment problem was given.

Taking into account the remarkable difference in the construction of the solution of both cases, the even number and the odd number of data, we introduce first the matrices which

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appear in the FMI for the even case (scalar version).

Definition 0.1 *Let $k = 2n+1$. Using the moments $s_0, s_1, \dots, s_{2n+1}$ we construct the following matrices*

$$\begin{aligned}
\tilde{K}_1 &= \{s_{j+k}\}_{j,k=0}^{n},\ \tilde{K}_2 = \{s_{j+k+1}\}_{j,k=0}^{n} \\
T &= \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}}_{n+1}, \\
v &= column[1, 0, \dots, 0] \\
u &= column[-s_0, -s_1, \dots, -s_n], \\
R_T(z) &= (I - zT)^{-1}. \\
K_1 &= -a\tilde{K}_1 + \tilde{K}_2, \quad K_2 = b\tilde{K}_1 - \tilde{K}_2, \\
u_1 &= u - aTu, \quad u_2 = -u + bTu.
\end{aligned}$$

Further, we introduce two auxiliary holomorphic functions

$$\begin{aligned}
\tilde{s}_1(z) &= (z-a)s(z), \\
\tilde{s}_2(z) &= (b-z)s(z), z \in \mathbb{C} \setminus [a,b].
\end{aligned} \tag{0.12}$$

Where $s(z)$ is the Stieltjes transform of σ : $\sigma \in \mathcal{M}[a,b]$.

In a similar way we introduce the matrices for the Potapov's FMI odd case.

Definition 0.2 *Let $k = 2n$. Let $T_1 = T$, T is defined in definition (0.1). Using the moments $s_0, s_1, \dots, s_{2n}$ we construct the following matrices*

$$\begin{aligned}
K_1 &= \{s_{j+k}\}_{j,k=0}^{n}, \\
v_1 &= column[1, 0, \dots, 0], \\
u_1 &= column[0, -s_0, \dots, -s_{n-1}], \\
R_{T_1}(z) &= (I - zT_1)^{-1}, \\
\tilde{K}_1 &= \{s_{j+k+1}\}_{j,k=0}^{n-1}, \\
\tilde{K}_2 &= \{s_{j+k}\}_{j,k=0}^{n-1} \\
\tilde{K}_3 &= \{s_{j+k+2}\}_{j,k=0}^{n-1}, \\
K_2 &= (a+b)\tilde{K}_1 - ab\tilde{K}_2 - \tilde{K}_3, \\
T_2 &= \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}}_{n}, \\
v_2 &= column[1, 0, \dots, 0], \\
R_{T_2}(z) &= (I - zT_2)^{-1}, \\
\tilde{u}_1 &= column[-s_0, -s_1, \dots, -s_{n-1}], \\
\tilde{u}_2 &= column[0, -s_0, \dots, -s_{n-2}] \\
\tilde{u}_3 &= [-s_1, -s_2, \dots, -s_n], \\
u_2 &= (a+b)\tilde{u}_1 - ab\tilde{u}_2 - \tilde{u}_3.
\end{aligned}$$

Here $u_1, v_1 \in \mathbb{R}^{n+1}$, $u_2, v_2 \in \mathbb{R}^n$. I represents the identity matrix of respective dimension. Further, we introduce two auxiliary holomorphic functions

$$\begin{aligned}
\tilde{s}_1(z) &= s(z), \\
\tilde{s}_2(z) &= (b-z)(z-a)s(z) - s_0 z, \\
& \quad z \in \mathbb{C} \setminus [a,b].
\end{aligned} \tag{0.13}$$

Where $s(z)$ is the Stieltjes transform of a non-negative σ on $[a,b]$.

We define the system of Potapov's FMI for the even and odd cases [2],[3].

Definition 0.3 *Let $(s_j)_{j=0}^{k}$ be a sequence of real numbers. The function s is called a solution of the associated system of V.P. Potapov's fundamental matrix inequality, if s satisfies the following properties:*

(i) s is holomorphic in $\mathbb{C} \setminus [a,b]$.

(ii) For $r \in \{1,2\}$ the inequality

$$\left[\begin{array}{c|c} K_r & R_{T_r}(z)\,[\,v_r \tilde{s}_r(z) - u_r\,] \\ \hline * & \{\tilde{s}_r(z) - \tilde{s}_r^*(z)\}/\{z - \bar{z}\} \end{array} \right] \geq 0 \tag{0.14}$$

holds.

Where K_r, T_r, u_r, $s_r(z)$ and v_r are defined as in (0.12) and (0.13). $*$ means the complex conjugate of $R_{T_r}(z)\,[\,v\tilde{s}_r(z) - u_r\,]$.

In definition 0.3 the auxiliary functions $\tilde{s}_r(z)$, $r \in \{1,2\}$ and T_k correspond to the even and

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odd cases of definition 0.1 and 0.2. Remark that for $k = 2n+1$ the matrix $T_1 = T_2 = T$ and $v_1 = v_2 = v$.

It turns out that the treatment of the matrix moment problem is equivalent to finding all solutions of corresponding fundamental matrix inequalities system of Potapov type (see [2],[3]):

Theorem 0.1 *The function $s(z)$ is a Stieltjes transform of $\sigma : \sigma \in \mathcal{M}([a,b],\{s_j\}_{j=0}^{k})$ iff $s(z)$ is a solution of the system of Potapov's Fundamental Matrix Inequalities (0.14).*

This theorem takes place for both the even and the odd case of data. In this way the problem of finding the Stieltjes transform of σ is reduced to the problem of finding holomorphic functions $s(z)$ (see definition 0.3) such the inequality (0.14) holds. Let $\mathcal{R}([a,b],(s_j)_{j=0}^{k})$ the set of Stieltjes transforms of $\mathcal{M}([a,b],\{s_j\}_{j=0}^{k})$.

Now we show that the SNC problem can be formulated in terms of a classical $[0,\theta]$–Hausdorff moment problem.

1 From the SNC problem to the classical Hausdorff moment problem.

The solution of the system (0.1) can be written in the following form:

$$x(\theta) = e^{A\theta}\left(x_0 + \int_0^\theta e^{-A\tau} b u(\tau) d\tau\right). \quad (1.1)$$

From the complete controllability of (0.1) there exists θ such that $x(\theta) = 0$.
Taking into account the relation

$$e^{-A\tau} b = \begin{pmatrix} 1 \\ -\tau \\ \vdots \\ \frac{(-1)^{n-1}}{(n-1)!}\tau^{n-1} \end{pmatrix}, \quad (1.2)$$

the equality (1.1) can be written in the form $-x_0^j = \frac{(-1)^{j-1}}{(j-1)!}\int_0^\theta \tau^{j-1}u(\tau)d\tau,\ j \in \{1,\ldots,n\}$. We write the last relation in an equivalent form

$$\begin{aligned} (-1)^j(j-1)!x_0^j &= 2\int_0^\theta \tau^{j-1}\frac{(u(\tau)+1)}{2}d\tau \\ &\quad - \int_0^\theta \tau^{j-1}d\tau, \\ \frac{\theta^j + (-1)^j j! x_0^j}{2j} &= \int_0^\theta \tau^{j-1} f(\tau) d\tau, \\ &\quad j \in \{1,\ldots,n\}. \end{aligned} \quad (1.3)$$

Thus, the SNC problem is reduced to the problem of finding the minimal θ and a function $0 \le f(\tau) \le 1$, $\tau \in [0,\theta]$ for which the relation (1.3) takes place.

Denote through $c_{j-1}(\theta, x_0)$, $j \in \{1,\ldots,n\}$ the left hand side of (1.3). Using the relation (0.11) for $L = 1$, $a = 0$, $b = \theta$, we obtain the data moments of the classical $[0,\theta]$–Hausdorff moment problem, which we symbolize through $s_j(\theta, x_0)$, $j \in \{0,\ldots,n\}$.

From the relation (0.11) we obtain the following

Proposition 1.1 *(See [5], pag. 324) The L–Markov moment problem with $c_{j-1}(\theta,x_0)$, $j \in \{1,\ldots,n\}$ entries is solvable iff the $[0,\theta]$–Hausdorff moment problem with entries $s_j(\theta,x_0)$, $j \in \{0,\ldots,n\}$ is solvable.*

In the next section we are going to show that the SNC problem is reduced to the problem of finding a solution of the Potapov's FMI (0.14).

2 Solution of the SNC problem.

Using the sequence $\{s_j(\theta,x_0)\}_{j=0}^{n}$, we construct Hankel matrices K_1, K_2 for the even as well as the odd number of data, and vectors

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u_r, v_r, $r = \{1, 2\}$ as described in definition 0.1 and 0.2.
We assume that K_1 and K_2 are positive definite, i.e. $\det K_r \neq 0$, $r = \{1, 2\}$. Observe that the case $\det K_1 = 0$ and/or $\det K_2 = 0$ corresponds to the time optimal control for the system (0.1).

Following the Potapov schema, we introduce a polynomial 2×2 matrix function (see [2], [3]), the so called *resolvent matrix* of the HMP. In the even case we define,

$$\begin{aligned}
U_{11}(z) &:= 1 - z u_2^* R_{T^*}(z) K_2^{-1} v,\\
U_{12}(z) &:= u_1^* R_{T^*}(z) K_1^{-1} u_1,\\
U_{21}(z) &:= -(\theta - z) z v^* R_{T^*}(z) K_2^{-1} v,\\
U_{22}(z) &:= 1 + z v^* R_{T^*}(z) K_1^{-1} u_1.
\end{aligned}$$

In odd case we define

$$\begin{aligned}
U_{11}(z) &:= 1 - z u_1^* R_{T_1^*}(z) K_1^{-1} v_1,\\
U_{12}(z) &:= M - z u_1^* R_{T_1^*}(z) K_1^{-1} v_1 M\\
&\quad + z u_1^* R_{T_1^*}(z) K_1^{-1} u_1,\\
U_{21}(z) &:= -z v_1^* R_{T_1^*}(z) K_1^{-1} v_1,\\
U_{22}(z) &:= 1 - z v_1^* R_{T_1^*}(z) K_1^{-1} v_1 M\\
&\quad + z v_1^* R_{T_1^*}(z) K_1^{-1} u_1.
\end{aligned}$$

Where $M = (1 + \theta[u_1^* K_1^{-1} v_1 - u_2 K_2^{-1} v_2])(\theta v_1^* K_1^{-1} v_1)^{-1}$.
We introduce two classes of functions which are set of parameters of solutions of the HMP.

Definition 2.1 *Let $\mathcal{R}[a, b]$ denote the class of all functions $w : \mathbb{C} \setminus [a, b] \to \mathbb{C}$ such that: w is holomorphic in $\mathbb{C} \setminus [a, b]$, $Im\ w(z) \geq 0$, $w(x) \geq 0$ if $x \in (-\infty, 0)$ and $w(x) \leq 0$ if $x \in (\theta, \infty)$.*

Definition 2.2 *Let $\mathcal{S}[a, b]$ denote the class of all functions $w : \mathbb{C} \setminus [a, b] \to \mathbb{C}$ such that: w is holomorphic in $\mathbb{C} \setminus [a, b]$, $Im\ w(z) \geq 0$, $w(x) \geq 0$ if $x \in (-\infty, 0) \cup (\theta, \infty)$.*

The next two theorems are concerned with the integral representation of matrix functions belonging to $\mathcal{R}[a, b]$ and $\mathcal{S}[a, b]$.

Theorem 2.1 *The following statement holds: $w \in \mathcal{R}[a, b]$ iff $w(z) = \int\limits_a^b (x - z)^{-1} \sigma(dx)$, where σ is a nonnegative measure on $[a, b]$.*

Theorem 2.2 *The following statement holds: $w \in \mathcal{S}[a, b]$ iff $w(z) = (b - z) \int\limits_a^b (x - z)^{-1} \sigma(dx)$, where σ is a nonnegative measure on $[a, b]$.*

The proofs of these theorems are available in [4].
The next theorem describes the set of solutions of the HMP (see [2],[3],[4]):

Theorem 2.3 *The fractional linear transformation*

$$s := \frac{U_{11} + w\, U_{12}}{U_{21} + w\, U_{22}} \tag{2.1}$$

yields a bijection between:
a) (In the even case), the parameter $w \in \mathcal{R}[a, b] \cup \infty$ and the Stieltjes transform $s \in \mathcal{R}([a, b], (s_j)_{j=0}^{2n+1})$.
b)(In the odd case), the parameter $w \in \mathcal{S}[a, b] \cup \infty$ and the Stieltjes transform $s \in \mathcal{R}([a, b], (s_j)_{j=0}^{2n})$.

The theorem 2.3 says that for a given parameter w (one can use the integral representation) we obtain the Stieltjes transform s of σ. To calculate σ (that is f, consequently the control u) from s, we use the Stieltjes–Perron inverse formula (we assume $\sigma(0) = 0$, $t \in (0, \theta]$):

$$\sigma(t) = \lim_{\epsilon + 0} \frac{1}{\pi} \int_0^t \mathrm{Im}\ s(x + i\epsilon) dx. \tag{2.2}$$

Taking into account the relation (0.9) and (2.2) we obtain $f(t)$, $t \in [0, \theta]$ from the relation :

$$f(t) = -\lim_{\epsilon \to +0} \arg\left((t + i\epsilon) s(t + i\epsilon)\right) \tag{2.3}$$

Consequently, the set of controls $U = \{u : u(t) = 2f(t) - 1,\ t \in [0, \theta]\}$ is the solution of the SNC problem.
Example. Consider the system $\dot{x}_1 = u$, $\dot{x}_2 = x_1$, $|u| \leq 1$, with initial position $x_1^0 = 0$, $x_2^0 = 1$. For $\theta = 3$ the matrices K_1 and K_2 are positive definite. In this case the solution of the equivalent Hausdorff moment problem is given by (2.1), where $U_{11} = 1 - \frac{12}{13} z$, $U_{12} = \frac{23}{15} - \frac{4}{5} z$, $U_{21} = \frac{1}{13} z(-31 + 12z)$, $U_{22} = 1 - \frac{41}{15} z - \frac{4}{5} z^2$.

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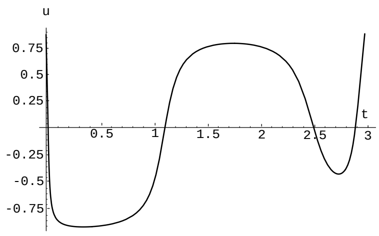
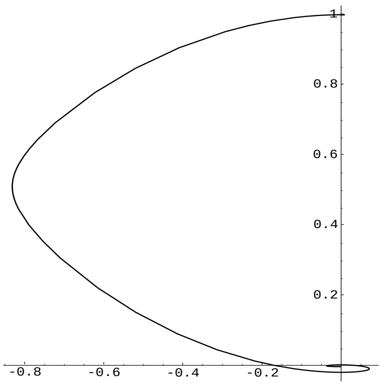
We set $w = (\theta - z)\int_0^\theta (t-z)^{-1}dt$, the correspondent measure $\sigma(t) = t$. The control $u(t) = 2f(t) - 1$ where,

$$f(t) = \begin{cases} \frac{1}{\pi}\left(h(t) - \frac{1+(-1)^k}{2}\pi\right), & t_k \le t < t_{k+1},\ k=0,..,3, \\ \frac{1}{\pi}(h(t) - \pi), & t_4 \le t \le 3. \end{cases}$$

$h(t) = \arctan\frac{38025\pi}{g(t)}$, $g(t) = 194435 + 3t(2704(-7+3t) + 75\pi^2(-3+t)(-31+12t)(-13+12t)) + \frac{15}{4}\lg\left|\frac{t-3}{t}\right|(-26(-3+2t)(65+144(-3+t)t) + 15(-3+t)t(-31+12t)(-13+12t)\lg\left|\frac{t-3}{t}\right|)$. $t_0 = 0, t_1 = 0.0197331, t_2 = 1.11521806, t_3 = 2.526024, t_4 = 2.9091237$. We obtain the following graphs.
Graph of the control:



Graph of the positional trajectory beginning at $x_0 = (0,\ 1)^T$. The vertical axis corresponds to the behavior of $x_2(t)$, the horizontal to $x_1(t)$.



Conclusion. The SNC problem was first reduced to a Markov moment problem. Due to the relation (0.9), the last problem was reduced to Hausdorff moment problem on $[0, \theta]$. By virtue of theorem 0.1 the problem of finding the Stieltjes transform of solution (a measure σ) of the Hausdorff moment problem was "translated" to the problem of finding a solution (a holomorphic in $\mathbb{C}\setminus[a,b]$ function) of the FMI (0.14). In the nondegenerate case (K_1, K_2 are positive definite), using the Stieltjes–Perron inverse formula, the solution of the SNC problem is given. It means, we have given (using a parameter) the solution set of the null controllability problem for the chain of integrators system except for the case $\det K_1$ and/or $\det K_2$ (denegerate case), which corresponds to the time optimal control (see[1]).

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